7.2 Quadratic Forms.

Consider the following functions of two and three Variables:

a) $$Q(x_1,x_2) = 5x_1^2 - 4x_1x_2 + 8x_2^2 \qquad (1.1)$$

b) $$Q(x_1,x_2,x_3) = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3 \qquad (1.2)$$

These types of functions are very common in applications of mathematics.

For example, (a) represent a Surface in $\mathbb{R}^3$

Also, if $Q(x_1,x_2) = C$ (constant), Q represents a "Conic curve" in $\mathbb{R}^2$.

They are particular cases of "Quadratic Forms" in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. In fact,

Quadratic Form in $\mathbb{R}^2$:

$$Q(x_1,x_2) = a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 \qquad (1.3)$$

Quadratic Form in $\mathbb{R}^3$:

$$Q(x_1,x_2,x_3) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3. \qquad (1.4)$$

The terms: $2a_3 x_1 x_2$ and

$2a_4 x_1 x_2$, $2a_5 x_1 x_3$, $2a_6 x_2 x_3$

are called <u>Cross products terms</u>.

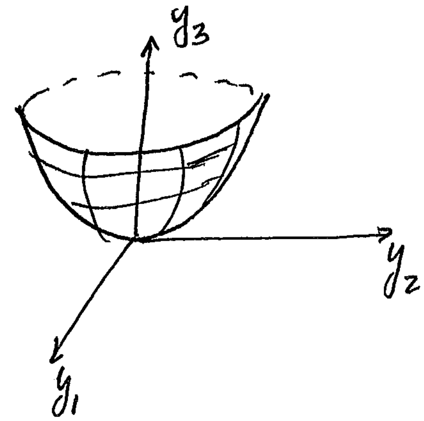
Also,

$$Q(y_1, y_2) = 4y_1^2 + 9y_2^2 \qquad (2.1)$$

and

$$Q(x_1, x_2, x_3) = -3y_2^2 + 3y_3^2 \qquad (2.2)$$

are quadratic forms. However, they do not have Cross products terms. In general, it is easier to work with quadratic forms without Cross product terms. For example, equation (2.1) is a well-known Surface in $\mathbb{R}^3$. It is a paraboloid.



Now, we will use the theory of Orthogonal diagonalization to reduce a more complicated quadratic form as (1.1) or (1.2) to a simpler one as (2.1) and (2.2).

I) Expressing Quadratic Forms by Symmetric Matrices.

The quadratic form (1.1)

$$Q(x_1,x_2) = 5x_1^2 - 4x_1x_2 + 8x_2^2$$

Can be expressed as

$$Q(x_1,x_2) = [x_1 \ \ x_2]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}^T A \vec{x}$$

where $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Since

$$[x_1 \ x_2]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \ x_2]\begin{bmatrix} 5x_1 - 2x_2 \\ -2x_1 + 8x_2 \end{bmatrix} =$$

$$= 5x_1^2 - 2x_2x_1 - 2x_1x_2 + 8x_2^2$$

$$= 5x_1^2 - 4x_1x_2 + 8x_2^2$$

$$= Q(x_1,x_2).$$

Similarly, the quadratic form:

$$Q(x_1,x_2,x_3) = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$$

Can be expressed as

$$[x_1\ x_2\ x_3]\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x}^T A \vec{x}$$

where

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since

$$[x_1\ x_2\ x_3]\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1\ x_2\ x_3]\begin{bmatrix} x_1 - 2x_2 \\ -2x_1 + 2x_3 \\ 2x_2 - x_3 \end{bmatrix}$$

$$= x_1^2 - 2x_2x_1 - 2x_1x_2 + 2x_3x_2 + 2x_2x_3 - x_3^2$$

$$= x_1^2 - 4x_1x_2 + 4x_2x_3 - x_3^2 = Q(x_1,x_2,x_3).$$

In general, any Quadratic form in $\mathbb{R}^3$ given by

$$Q(x_1,x_2,x_3) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2$$

$$+ 2a_4\, x_1x_2 + 2a_5\, x_1x_3 + 2a_6\, x_2x_3$$

Can be expressed using a Symmetric matrix as

$$[x_1\ x_2\ x_3] \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (3.2.1)$$

Notice that this is a Symmetric matrix.

Since (3.2.1) is equal to

$$[x_1\ x_2\ x_3] \begin{bmatrix} a_1x_1 + a_4x_2 + a_5x_3 \\ a_4x_1 + a_2x_2 + a_6x_3 \\ a_5x_1 + a_6x_2 + a_3x_3 \end{bmatrix}$$

$$= \underline{a_1x_1^2} + a_4x_2x_1 + a_5x_3x_1 + a_4x_1x_2 + \underline{a_2x_2^2} + a_6x_3x_2$$

$$+ a_5x_1x_3 + a_6x_2x_3 + \underline{a_3x_3^2}$$

$$= a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3.$$

$$= Q(x_1,x_2,x_3).$$

II) Reducing Quadratic Form (1.1) and (1.2) to (2.1) and (2.2), respectively, using orthogonal diagonalization.

If there are not cross product terms the matrix expressions of the quadratic forms are given by means of diagonal matrices.

In fact,

$$Q(y_1, y_2) = 4y_1^2 + 9y_2^2 = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and

$$Q(y_1, y_2, y_3) = -3y_2^2 + 3y_3^2 = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Elimination of the Cross-Product terms in a Quadratic Form

A general quadratic form in $\mathbb{R}^n$ can be expressed as

$$Q(x_1, \dots, x_n) = \vec{x}^T A \vec{x},$$

where $A_{n \times n}$ matrix and A is symmetric.

Since A is Symmetric, there exists an orthonormal matrix P Such that

$$\boxed{A = PDP^T} \quad \text{or} \quad \boxed{P^TAP = D}$$

where D is a diagonal matrix with the eigenvalues of A along the diagonal and the columns of P are normalized eigenvectors corresponding to the eigenvalues of A.

Therefore, if we introduce the change of Variables

$$\boxed{\vec{x} = P\vec{y}}$$

$$Q(x_1, x_2, \ldots, x_n) = \vec{x}^T A \vec{x} = (P\vec{y})^T A (P\vec{y})$$
$$= \vec{y}^T(P^TAP)\vec{y} = \vec{y}^T D \vec{y}.$$

Or

$$Q(x_1, x_2, \ldots, x_n) = Q(y_1, y_2, \ldots, y_n) = \vec{y}^T D \vec{y}$$

$$= \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2.$$

With no Cross product terms.

Theorem (Principal axes theorem)

$A_{n\times n}$ Symmetric

then, there is an orthogonal change of variable

$$\vec{x} = P\vec{y}$$

that transforms the quadratic form

$$\vec{x}^T A \vec{x}$$

into a quadratic form

$$\vec{y}^T D \vec{y}, \qquad D \text{ diagonal.}$$

With no cross-product term.

The diagonal entries of D are the eigenvalues of A

The columns of P are the corresponding eigenvectors (normalized).

they are also called the principal axes of the quadratic form.

The vector $\vec{y}$ (new variables) is the coordinate vector of $\vec{x}$ relative to the orthonormal basis defined by the columns of P or principal axes.

Transformation of Quadratic form (1.1) and (1.2) into (2.1) and (2.2), respectively, using Ppal Axes Theorem.

It can be easily proved that

for $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$

Eigenvalues — Eigenvectors (normalized)

$\lambda_1 = 4 \longrightarrow \vec{u}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

$\lambda_2 = 9 \longrightarrow \vec{u}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Therefore,

$$P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

Then,

$$Q(x_1, x_2) = \vec{x}^T A \vec{x} = \underline{5x_1^2 - 4x_1x_2 + 8x_2^2}$$

$$= \vec{y}^T D \vec{y} = [y_1 \ y_2] \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underline{4y_1^2 + 9y_2^2}.$$

In general, Quadratic forms $Q(x_1, x_2)$ are used to define Conic Sections in the plane. They are:
Circles, ellipses, and hyperbolas, but not parabolas.
They correspond to an equation of the form:

$$Q(x_1, x_2) = c \text{ (constant)}$$

For instance in our previous example,

$$Q(x_1,x_2) = 36 \iff 5x_1^2 - 4x_1x_2 + 8x_2^2 = 36$$

By introducing the change of Variable

$P\hat{y} = \check{x}$ more precisely $\begin{array}{cc} \hat{u}_1 & \hat{u}_2 \end{array}$ $\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q(x_1,x_2) = 36$ reduces to

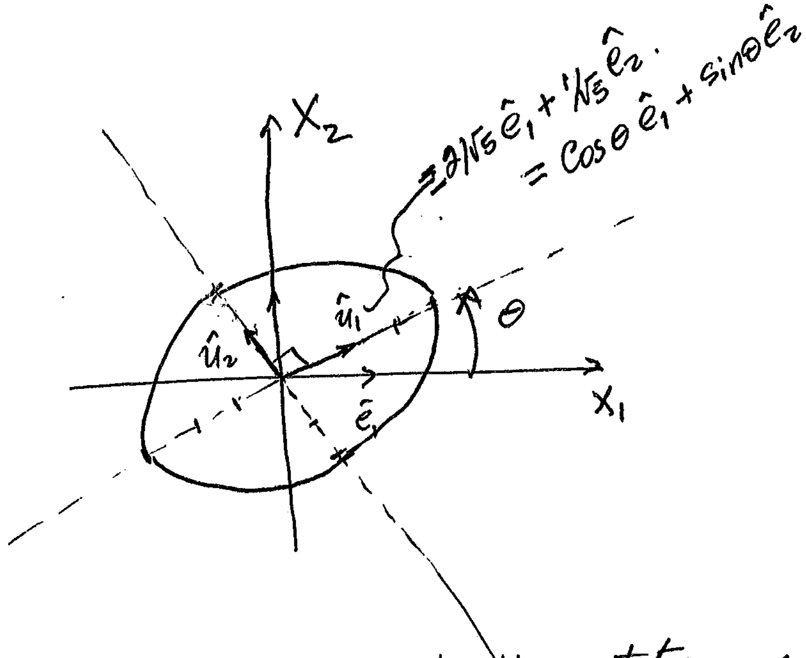
$$Q(y_1,y_2) = 4y_1^2 + 9y_2^2 = 36$$

or $\boxed{\dfrac{y_1^2}{9} + \dfrac{y_2^2}{4} = 1}$ (ellipse in standard position) in the y_1y_2-plane

Therefore, the equation

$$5x_1^2 - 4x_1x_2 + 8x_2^2 = 36$$

Corresponds to an ellipse in the original Variables x_1 and x_2 rotated by certain angle θ, which can be determined by the entries in the columns of P.



The Cross-product is due to the rotation of the ellipse from standard position in the x_1x_2-plane.

Since, $\cos\theta = 2/\sqrt{5}$ and $\sin\theta = 1/\sqrt{5}$

$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{2} \Rightarrow \theta \simeq 26.6^\circ$$

<u>Transformation of (1.2) into (2.2)</u>, <u>Using Ppal axes theorem</u>

It is easily proved that the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

are

$$\lambda_1 = 0,\ \hat{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix},\quad \lambda_2 = -3,\ \hat{u}_2 = \begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$\lambda_3 = 3,\ \hat{u}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

Therefore according to the "Ppal axes theorem", the change of Variable

$$\vec{x} = P\vec{y},$$

where
$$P = \begin{bmatrix} 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

reduces $Q(x_1, x_2, x_3)$ to

$$Q(y_1, y_2, y_3) = [y_1 \ y_2 \ y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} =$$

$$= -3y_1^2 + 3y_3^2 \quad \checkmark$$

Clasification of Quadratic Forms

Use book. graphs.

Proof of Theorem 5

By "Ppal Axes Theorem" for any quadratic form $Q(x_1, .., x_n)$, there exists an orthogonal change of variable

$\vec{x} = P\vec{y}$ s.t.
$$Q(x_1, x_2 .., x_n) = \vec{x}^T A \vec{x} = \vec{y}^T D \vec{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

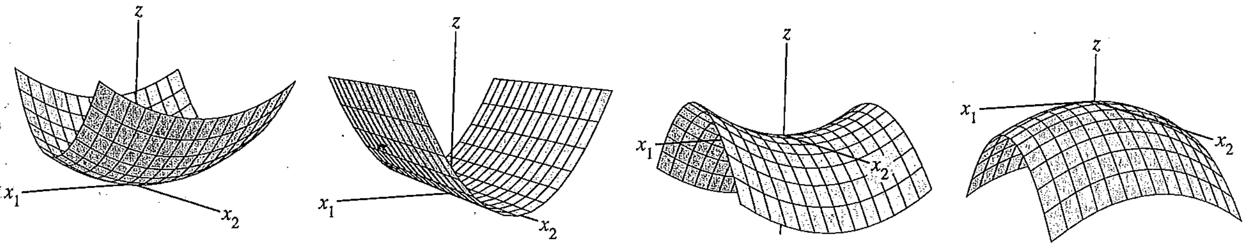
where $\lambda_1 .., \lambda_n$ eigenvalues of A

One-to-One Correspondence $\vec{x} = (x_1 .., x_n) \longleftrightarrow \vec{y} = (y_1, .., y_n)$
allows to prove theorem easily.

Let $P = [\,\mathbf{u}_1 \;\; \mathbf{u}_2\,] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Then P orthogonally diagonalizes A, so the change of variable $\mathbf{x} = P\mathbf{y}$ produces the quadratic form $\mathbf{y}^T D\mathbf{y} = 3y_1^2 + 7y_2^2$. The new axes for this change of variable are shown in Fig. 3(a). ■

Classifying Quadratic Forms

When A is an $n \times n$ matrix, the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is a real-valued function with domain $\mathbb{R}^n$. Figure 4 displays the graphs of four quadratic forms with domain $\mathbb{R}^2$. For each point $\mathbf{x} = (x_1, x_2)$ in the domain of a quadratic form Q, the graph displays the point (x_1, x_2, z) where $z = Q(\mathbf{x})$. Notice that except at $\mathbf{x} = \mathbf{0}$, the values of $Q(\mathbf{x})$ are all positive in Fig. 4(a) and all negative in Fig. 4(d). The horizontal cross-sections of the graphs are ellipses in Figs. 4(a) and 4(d) and hyperbolas in Fig. 4(c).



(a) $z = 3x_1^2 + 7x_2^2$ (b) $z = 3x_1^2$ (c) $z = 3x_1^2 - 7x_2^2$ (d) $z = -3x_1^2 - 7x_2^2$

FIGURE 4 Graphs of quadratic forms.

The simple 2×2 examples in Fig. 4 illustrate the following definitions.

DEFINITION

A quadratic form Q is:

a. **positive definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,

b. **negative definite** if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$,

c. **indefinite** if $Q(\mathbf{x})$ assumes both positive and negative values.

Also, Q is said to be **positive semidefinite** if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, and to be **negative semidefinite** if $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x}$. The quadratic forms in parts (a) and (b) of Fig. 4 are both positive semidefinite, but the form in (a) is better described as positive definite.

Theorem 5 characterizes some quadratic forms in terms of eigenvalues.

THEOREM 5

Quadratic Forms and Eigenvalues

Let A be an $n \times n$ symmetric matrix. Then a quadratic form $\mathbf{x}^T A\mathbf{x}$ is:

a. positive definite if and only if the eigenvalues of A are all positive,

b. negative definite if and only if the eigenvalues of A are all negative, or

c. indefinite if and only if A has both positive and negative eigenvalues.